A Dirac-dual Dirac method in periodic cyclic homology

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Take a real Lie group G and K a maximal compact subgroup of it. The associated homogeneous space G/K is a contractible Euclidian space. Suppose G is such that G/K is even-dimensional.

Theorem *(Connes-Kasparov) There exists an isomorphism*

$$K_\bullet(C_r^\star(K)) \xrightarrow{\sim} K_\bullet(C_r^\star(G)).$$

The question of this talk treats whether this isomorphism also holds at the smooth homological level. Namely, does there exist an isomorphism between the periodic cyclic homology of the associated smooth convolution algebras

$$HP_\bullet(\mathcal{C}^\infty(K), \star) \xrightarrow{\sim} HP_\bullet(\mathcal{C}_c^\infty(G), \star)\ ? \tag{1}$$

To get such an isomorphism, we would like to establish some smooth analogue of the Dirac-dual Dirac method, usually defined at the level of bivariant K-theory to fit the periodic cyclic homology framework. Let us discuss the implementation of this method at the bivariant K-theoretical level.

The Connes-Kasparov isomorphism can be thought as a two-step isomorphism

$$\begin{tikzcd} K_\bullet(C_r^\star(K)) \arrow[r, "\sim"] & K_\bullet(\mathcal{C}_0(G/K) \rtimes_r G) \arrow[rr, bend left, "\alpha"] & & K_\bullet(C_r^\star(G)) \arrow[ll, bend left, "\beta"] \end{tikzcd}$$

where the first-step is a consequence of a Morita-Rieffel equivalence of $C^\star$-algebras and the second is the purpose of the Dirac-dual Dirac method. The idea is to define some homomorphisms of abelian groups α and β that are inverse to each others. As morphisms from K-group to K-group they might come from bivariant classes that respect group actions, namely equivariant KK-theory classes. The fact that their composition is the identity can be translated to the triviality of their Kasparov products in the bivariant K-theory framework.

Theorem *(Kasparov) There exists classes $\alpha \in KK_G(\mathcal{C}_0(G/K), \mathbb{C})$ and $\beta \in KK_G(\mathbb{C}, \mathcal{C}_0(G/K))$ called **Dirac** and **dual Dirac** elements whose Kasparov products verify:*

$$\alpha \times \beta = 1 \ \textit{ and } \ Res_K^G(\beta \times \alpha) = 1.$$

This theorem implements a Dirac-dual Dirac method in bivariant K-theory and is mainly responsible of the Connes-Kasparov isomorphism.

Let us try to propose a smooth analogue of this method to fit periodic cyclic homology. First, we can also draw the equation

$$\begin{tikzcd} HP_\bullet(\mathcal{C}^\infty(K), \star) \arrow[r, "\sim"] & HP_\bullet(\mathcal{C}_c^\infty(G/K) \rtimes G) \arrow[rr, bend left, "\alpha"] & & HP_\bullet(\mathcal{C}_c^\infty(G), \star) \arrow[ll, bend left, "\beta"] \end{tikzcd} \tag{2}$$

where the left-hand side isomorphism is obtained as a consequence of the notion of quasi-Morita equivalence developed in my thesis [Gas26, §2.11]. To get the right-hand side isomorphism, we want to define some Dirac and dual Dirac elements in bivariant equivariant periodic cyclic homology that respect some similar properties as before. The Dirac class α is not only an equivariant KK-class but comes from an elliptic operator over the manifold G/K. Namely, $\alpha \in KK_G(\mathcal{C}_0(G/K), \mathbb{C})$ can be represented as a tuple

$$\alpha = \left[\left(\mathcal{H} = \mathcal{L}^2(\Omega^\star(G/K)), \rho = \text{Clifford mult.}, F = \Delta(1+\Delta^2)^{-1/2}, \pi = \text{left transl.}\right)\right]$$

where the different commutators between these operators belongs to $\mathcal{K}(\mathcal{H})$ by definition of a Kasparov bimodule.

Lemma *We can deform α so that it becomes p-summable over $\mathcal{H}$ for $p > \dim(G/K)$, i.e. so that the different commutators belong to*

$$\ell^p(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) \mid \mathrm{Tr}((T^\star T)^p) < \infty\} \subseteq \mathcal{K}(\mathcal{H}).$$

Corollary *The Dirac and dual Dirac elements induce equivariant classes in bivariant periodic cyclic homology*

$$\alpha \in HP_G(\mathcal{C}_c^\infty(G/K), \ell^{p,G}(\mathcal{H})) \text{ and } \beta \in HP_K(\mathbb{C}, \mathcal{C}_c^\infty(G/K))$$

where $\ell^{p,G}(\mathcal{H})$ is made of the G-smooth vector in the p-th Schatten ideal.

The main problem is that the apparition of this G-smooth Schatten ideal prevents to compose this classes using Kasparov product. Indeed, we are unable to compute the periodic cyclic homology of the codomain $\ell^{p,G}(\mathcal{H})$. We are able to prove that the p-th Schatten ideal $\ell^p(\mathcal{H})$ possesses trivial periodic cyclic homology but we still don't know anything about its subalgebra of G-smooth vectors. To solve this issue we chose to define a stabilization algebra that *absorbs* this error, namely a stable envelop of the error terms produced by the Dirac operator.

Theorem (*G., 2026*) *There exists an algebra $\mathcal{D}$ with property $\mathcal{D} \otimes \ell^{p,G}(\mathcal{H}) \simeq \mathcal{D}$ such that the Dirac and dual Dirac element induce classes*

$$\alpha \in HP_G(\mathcal{C}_c^\infty(G/K) \otimes \mathcal{D}, \mathcal{D}) \ \text{ and } \ \beta \in HP_K(\mathcal{D}, \mathcal{C}_c^\infty(G/K) \otimes \mathcal{D})$$

which verify $\mathrm{Res}_K^G(\alpha) \times \beta = 1$ and $\beta \times \mathrm{Res}_K^G(\alpha) = 1$.

The definition of $\mathcal{D}$ comes from the following observation. One can compute the product $\beta \times \mathrm{Res}_K^G(\alpha)$ as an element of $HP_K(\mathbb{C}, \ell^{p,G}(\mathcal{H}))$ and it turns out to be represented by any algebra homomorphism $\mathbb{C} \longrightarrow \ell^{p,G}(\mathcal{H})$ of the form $1 \mapsto P$ where P is an isometric rank-one projection into a subspace of K-fixed vectors. Then one can built a *tower* of isometric inclusions $\ell^{p,G}(\mathcal{H})^{\otimes n} \longrightarrow \ell^{p,G}(\mathcal{H})^{\otimes n+1}, T \mapsto P \otimes T$ and take its inductive limit

$$\mathcal{D} := \overline{\lim_{\rightarrow} \ell^{p,G}(\mathcal{H})^{\otimes n}}.$$

Namely, the stabilization algebra is the completion for the induced norm of the associated inductive limit. It absorbs the defect of the Dirac and dual Dirac element in periodic cyclic homology to be inverse to each others.

At the level of periodic cyclic homology, the dual Dirac element is only defined K-equivariantly so the duality between these elements at the level of periodic cyclic homology is at most K-equivariant. Indeed, a theorem of Nistor [Nis93], [Gas26, §3.2] states that this is sufficient to get a G-equivariant global isomorphism. Together with the equation (2), we obtained an answer to the question (1), after stabilization by the algebra $\mathcal{D}$.

Corollary *The Dirac element $\alpha \in KK_G(\mathcal{C}_0(G/K), \mathbb{C})$ realizes the isomorphism*

$$HP_\bullet((\mathcal{C}^\infty(K), \star) \otimes \mathcal{D}) \simeq HP_\bullet((\mathcal{C}_c^\infty(G/K) \otimes \mathcal{D}) \rtimes G) \xrightarrow{\sim} HP_\bullet((\mathcal{C}_c^\infty(G), \star) \otimes \mathcal{D}).$$

Moreover, this method also provides some Connes-Kasparov theorem in periodic cyclic homology **with coefficients** without any restriction in the dimension.

Theorem (*G., 2026*) *Let G be a real Lie group with maximal compact subgroup K. For every complete locally convex algebra A endowed with a smooth action of G, the Dirac element $\alpha \in KK_G(\mathcal{C}_0(G/K), \mathbb{C})$ realizes the following isomorphism:*

$$HP_\bullet(\mathcal{A} \rtimes K) \simeq HP_\bullet(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathcal{A} \rtimes G).$$

where $\mathcal{A} = A \otimes_\pi \mathcal{D}$ is the stabilization of A.

References

[Gas26] Axel Gastaldi. "Periodic cyclic homology of smooth crossed product algebras". In: (2026). URL: https://axelgastaldi-math.com/Periodic%20cyclic%20homology%20of%20smooth%20crossed%20product%20algebras.pdf.

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